# Lect23-0412a Connected

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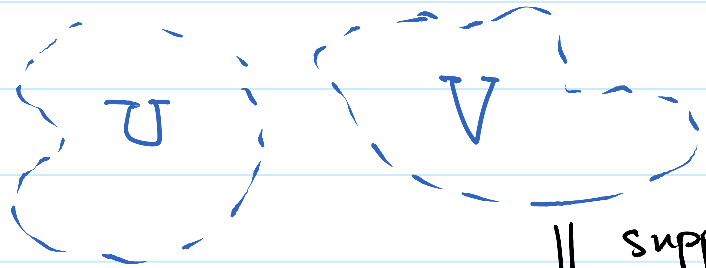
## More about Connected components

Suppose $X$ is disconnected

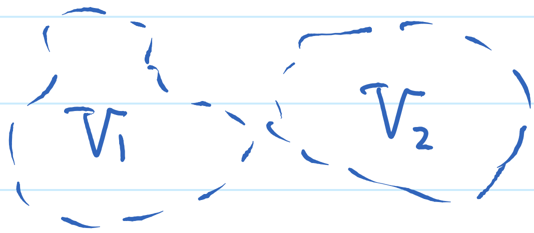
$\parallel$




$U, V$ both open & closed in $X$

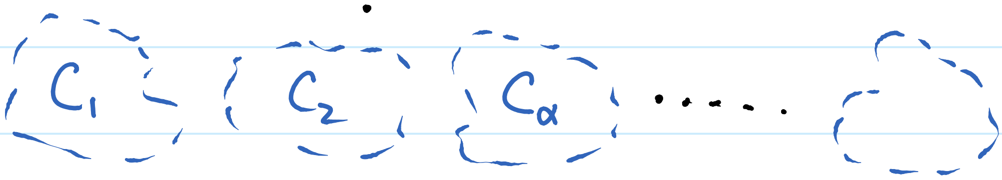
connected? Suppose yes

$\parallel$ suppose disconnected



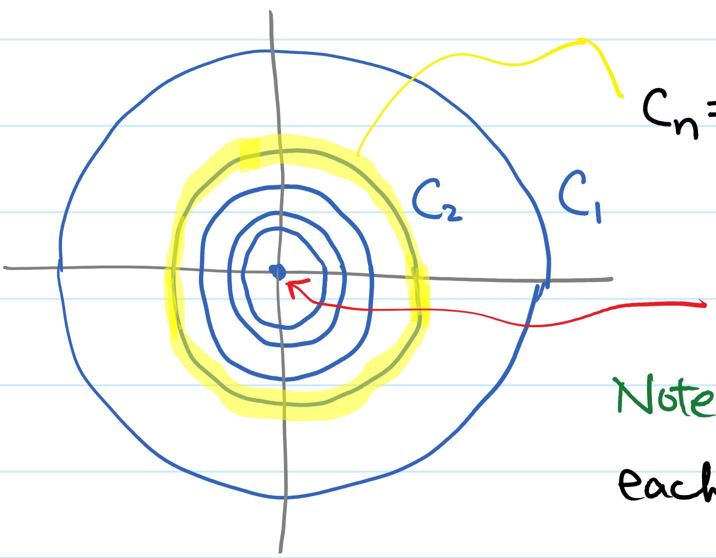

$V_1, V_2$ both open & closed in $V$ and in $X$

$\vdots$




Qu: Is each $C_\alpha$ both open and closed in $X$ ?

Example. $X = \bigcup_{0<n\in\mathbb{Z}} \underbrace{\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=\frac{1}{n^2}\right\}}_{C_n} \cup \underbrace{\{(0,0)\}}_{C_\infty}$




$C_n = (\text{open}) \cap X, \ 1 \le n \in \mathbb{Z}$

$\therefore$ open in $X$

$C_\infty$ is not open

Note. In this example, each $C_n$ is closed in $X$

# Connect to closure

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**Exercise.** If $X = C_1 \cup \cdots \cup C_n$ has finitely many connected components then each $C_k$ is both open and closed.

**Theorem** Let $A$ be a connected subset in $X$ and let $A \subset B \subset \bar{A}$. Then $B$ is connected

As a result, each connected component $C_\alpha$ is closed.

Because $\bar{C}_\alpha$ is connected and $\bar{C}_\alpha \supset C_\alpha$.

By maximality of $C_\alpha$, $\bar{C}_\alpha = C_\alpha$ and it is closed.

Proof.

Let $S \subset B$ be both open and closed in $B$

$\therefore$ $S = G \cap B = F \cap B$ where $G, X \setminus F \in \mathcal{J}$

$S \cap A = G \cap A = F \cap A$ is both open closed in $A$

$\therefore$ $\underset{\displaystyle G \cap A}{\underset{\shortparallel}{S \cap A}} = \phi$ or $\underset{\displaystyle F \cap A}{\underset{\shortparallel}{S \cap A}} = A$

$\therefore$ $A \subset \underbrace{X \setminus G}_{\text{closed}}$ $\qquad$ $\therefore$ $\underset{\text{closed}}{F} \supset A$

$\therefore$ $\underset{\displaystyle B}{\underset{\cup \text{ given}}{\bar{A}}} \subset X \setminus G$ $\qquad$ $\therefore$ $F \supset \underset{\displaystyle B}{\underset{\cup \text{ given}}{\bar{A}}}$

$\therefore S = G \cap B = \phi$ or $S = F \cap B = B$

Orthogonal

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**Example.** $O(n) = \{ n\times n \text{ orthogonal matrices} \} \subset \mathbb{R}^{n^2}$

$= \{ Q \in \mathbb{R}^{n^2} : Q^TQ = QQ^T = I \}$

Consider a function $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n(n+1)/2}$

$A \longmapsto \underbrace{A^TA}_{\text{symmetric}}$

continuous

**Qu.** What is $O(n)$ in terms of $f$?

$O(n) = f^{-1}(I)$

The pre-image of a continuous function

No conclusion about its connectedness.

Consider $g : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$

$A \longmapsto \det(A)$

Clearly, $g$ is continuous and

$g|_{O(n)} : O(n) \longrightarrow \{-1, 1\}$ is surjective

Thus, $O(n)$ is disconnected

**Qu.** What about $SO(n) = g^{-1}(1)$?

It is indeed path connected

A space $X$ is path connected if

$\forall\ x_0, x_1 \in X\ \exists$ continuous $\gamma : [0,1] \longrightarrow X$

such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

# Path Connected

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**Theorem** $X$ is path connected $\Rightarrow$ $X$ is connected

- The image $\gamma([0,1]) \subset X$ is connected
- $\forall\ x_0, x_1 \in X \quad x_0 \sim x_1$

Hence $X = [x_0]$, only one component

**Explore.** How to show $SO(n)$ is path connected?

Consider $U(n) = \{n \times n \text{ unitary matrices}\} \subset \mathbb{C}^{n^2}$

$$= \{A \in \mathbb{C}^{n^2} : A^*A = AA^* = I\}$$

Now, $g: \mathbb{C}^{n^2} \longrightarrow \mathbb{C} : M \longmapsto \det(M)$

is still continuous but the surjection is

$$g|_{U(n)} : U(n) \longrightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

**Cannot** conclude that $U(n)$ is disconnected.

**Exploration.**

(1) Show that $U(1)$ is the circle

(2) Show that $U(2)$ is homeomorphic to

quaternion $= \{a + bi + cj + dk : i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j\}$

(3) Find out why $U(n)$ is connected

# Locally Connected

**Qu.** Do you still remember how to define a local topological property?

A space $X$ is locally connected if at every $x \in X$, $\exists$ local base of connected nbhds.

That is, $\forall\, x \in X\ \exists\, \mathcal{U}_x \subset \mathcal{J}$ such that

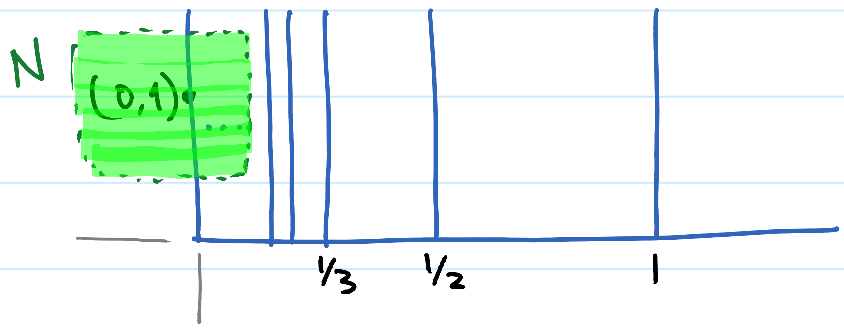
(i) every $U \in \mathcal{U}_x$ is connected

(ii) if $x \in \mathring{N}$ then $\exists\, U \in \mathcal{U}_x$, $x \in U \subset N$

**Qu.** Give an example of locally connected but disconnected space.

**Example.** Connected $\not\Longrightarrow$ Locally connected

Let $X = \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0,y) \in \mathbb{R}^2 : y \geq 0\}$
$\cup \{(\frac{1}{n}, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } 0 \leq n \in \mathbb{Z}\}$




$X$ is path connected, $\therefore$ connected

$(0,1) \in X$ has a nbhd $N \cap X$, which does not contain a connected nbhd of $(0,1)$